CALCULATION OF STREAMLINES WITH A KNOWN PRESSURE DISTRIBUTION ON THE SURFACE OF A RIGID BODY

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We shall consider the calculation of streamlines in an inviscid gas with a given pressure distribution on the surface of a rigid body. The problem reduces to the solution of a first order partial differential equation.

In connection with the difficulties of calculation of three-dimensional gas flows and the need for including in the input data the velocity components (or streamlines) on the external boundary for the calculation of the boundary layer, there is interest in the problem of calculating streamlines from the known pressure distribution on the surface of a rigid body, obtained, for example, by experiment *. In accordance with the assumptions of boundary layer theory we shall assume that the gas is inviscid, the bounday layer coinciding with the surface of the body, and that the flow is isoenergetic. We shall make use of an orthogonal curvilinear system of coordinates, such that coordinate lines q_1 and q_2 are located on the surface of the body. For the solution of the formulated problem we can make use of one of the two equations of Euler (projections of the acceleration at the surface of the body)

$$\frac{v_1}{H_1}\frac{\partial v_1}{\partial q_1}+\frac{v_2}{H_2}\frac{\partial v_1}{\partial q_2}+\frac{v_2}{H_1H_2}\left(v_1\frac{\partial H_1}{\partial q_2}-v_2\frac{\partial H_2}{\partial q_1}\right)=-\frac{1}{\rho H_1}\frac{\partial p}{\partial q_1} \qquad (1)$$

$$\frac{v_1}{H_1}\frac{\partial v_2}{\partial q_1}+\frac{v_2}{H_2}\frac{\partial v_2}{\partial q_2}-\frac{v_1}{H_1H_2}\left(v_1\frac{\partial H_1}{\partial q_2}-v_2\frac{\partial H_2}{\partial q_1}\right)=-\frac{1}{\rho H_2}\frac{\partial p}{\partial q_2} \qquad (2)$$

For example, we may use the first together with the formulas for the determination of pressure and density, arising from the second equation of Euler and the equation of conservation of energy

$$p=p_0\left(1-\frac{w^2}{w_m^2}\right)^{\frac{\gamma}{\gamma-1}}, \qquad \rho=\rho_0\left(1-\frac{w^2}{w_m^2}\right)^{\frac{1}{\gamma-1}} \qquad \left(w^2=v_1^2+v_2^2,\ w_m^2=\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\right) \qquad (3)\quad(4)$$

Here v_1 and v_2 are the projections of the velocity, H_1 and H_2 are the Lame coefficients, p_0 and ρ_0 are constant along the streamlines, and γ is the ratio of the specific heats.

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* A geometrical method of constructing the streamlines from a known pressure distribution is considered in a paper by Vaglio-Laurin [1].

The third equation of Euler and the equation of continuity cannot be used, since they contain the unknown derivatives along the normal to the surface of the body.

Let θ be the angle formed by the velocity w with the tangent to the curve q_1. If we substitute

$$v_1 = w\cos\theta, \qquad v_2 = w\sin\theta$$

in Equation (1) and make use of Expressions (3) and (4), we obtain

$$\frac{\cos\theta}{H_1}\frac{\partial\theta}{\partial q_1} + \frac{\sin\theta}{H_2}\frac{\partial\theta}{\partial q_2} = \tag{5}$$

$$= \frac{\gamma-1}{2\gamma}\frac{w_m^2 - w^2}{w^2}\frac{1}{H_1\sin\theta}\frac{\partial\ln p_0}{\partial q_1} - \frac{\sin\theta}{H_1}\frac{\partial\ln(H_2 w)}{\partial q_1} + \frac{\cos\theta}{H_2}\frac{\partial\ln(H_1 w)}{\partial q_2}$$

This equation can however also be obtained from (2), if we consider the condition for conservation of total head along a streamline

$$\frac{\cos\theta}{H_1}\frac{\partial p_0}{\partial q_1} + \frac{\sin\theta}{H_2}\frac{\partial p_0}{\partial q_2} = 0$$

Equation (5) is a first order partial differential equation for angle θ, and the right-hand side

$$R = \frac{\gamma-1}{2\gamma}\frac{(p/p_0)^\varkappa}{1-(p/p_0)^\varkappa}\left(\frac{\sin\theta}{H_1}\frac{\partial\ln p}{\partial q_1} - \frac{\cos\theta}{H_2}\frac{\partial\ln p}{\partial q_2}\right) -$$

$$-\frac{\sin\theta}{H_1}\frac{\partial\ln H_2}{\partial q_1} + \frac{\cos\theta}{H_2}\frac{\partial\ln H_1}{\partial q_2} \qquad \left(\varkappa = \frac{\gamma-1}{\gamma}\right)$$

depends on the coordinates q_1 and q_2 and the sought angle θ.

Equation (5) reduces to a system of ordinary differential equations

$$\frac{H_1\,dq_1}{\cos\theta} = \frac{H_2\,dq_2}{\sin\theta} = \frac{d\theta}{R} \tag{6}$$

whose characteristics are the streamlines. To integrate the system (6) it is necessary to know the value of the total head p_0 at the surface of the body and the angle θ on any curve which is not a characteristic (Cauchy's problem).

The problem is solved most simply for pointed body with the attached bow shock wave, since in this case we can assume that p_0 and θ are known on a closed curve on the surface of the body, located close to the point, which is not a characteristic. In the case of blunt body the value of p_0 is constant on its surface and is easily determined if the flow has two planes of symmetry or if, in the case of spherical nose, the region of subsonic flow is bounded by the spherical portion of the surface. In the latter case the angle θ is also known on a curve which is not a characteristic. If, however, the angle θ is known only on one streamline, i.e. on the characteristic streamline, and this is so, for example, in the case of flow with one plane of symmetry, then this is insufficient for the solution of the problem [2].

To obtain the missing initial data we can make use either of a second streamline on which the angle θ is known (a flow with two planes of symmetry), or try to obtain supplementary data on the first characteristic curve. Let us consider the coordinate curve q_1, located in the plane of symmetry of the flow, being at the same time a streamline. Differentiating (2) with respect to q_2 and bearing in mind that on this streamline

$$v_2 = \frac{\partial v_1}{\partial q_2} = \frac{\partial p}{\partial q_2} = \frac{\partial \rho}{\partial q_2} = \frac{\partial H_1}{\partial q_2} = 0$$

we obtain

$$\frac{H_2 w}{H_1}\frac{\partial^2 v_2}{\partial q_1\partial q_2} + \left(\frac{\partial v_2}{\partial q_2}\right)^2 + \frac{w}{H_1}\frac{\partial H_2}{\partial q_1}\frac{\partial v_2}{\partial q_2} - \frac{w^2}{H_1}\frac{\partial^2 H_1}{\partial q_2^2} + \frac{1}{\rho}\frac{\partial^2 p}{\partial q_2^2} = 0 \tag{7}$$

or, since on this curve $\partial\theta / \partial q_2 = 1 / w \partial v_2 / \partial q_2$, then

$$\frac{H_2}{H_1}\frac{\partial}{\partial q_1}\left(\frac{\partial\theta}{\partial q_2}\right)+\left(\frac{\partial\theta}{\partial q_2}\right)^2+\frac{H_2}{H_1}\frac{\partial\ln(H_2w)}{\partial q_1}\frac{\partial\theta}{\partial q_2}-\frac{1}{H_1}\frac{\partial^2H_1}{\partial q_2{}^2}+\frac{1}{\rho w^2}\frac{\partial^2p}{\partial q_2{}^2}=0 \qquad (8)$$

Integrating equation (8), we can find the value of $\partial\theta / \partial q_2$, needed for a solution of the system (6), in the case of a given angle θ on the characteristic curve. At this critical point $(w = 0)$

$$\left(\frac{\partial v_1}{\partial q_1}\right)^2=-\frac{1}{\rho}\frac{\partial^2p}{\partial q_1{}^2}, \qquad \left(\frac{\partial v_2}{\partial q_2}\right)^2=-\frac{1}{\rho}\frac{\partial^2p}{\partial q_2{}^2} \qquad (9)$$

Equations (5) and (8) have the simplest form in the case of a plane rigid surface $(H_1 = H_2 = 1,\ q_1 = x,\ q_2 = y)$

$$\cos\theta\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial\theta}{\partial y}=\frac{\gamma-1}{2\gamma}\frac{(p/p_0)^\varkappa}{1-(p/p_0)^\varkappa}\left(\sin\theta\frac{\partial\ln p}{\partial x}-\cos\theta\frac{\partial\ln p}{\partial y}\right)$$

$$\frac{\partial}{\partial x}\frac{\partial\theta}{\partial y}+\left(\frac{\partial\theta}{\partial y}\right)^2+\frac{\partial\ln w}{\partial x}\left(\frac{\partial\theta}{\partial y}\right)+\frac{1}{\rho w^2}\frac{\partial^2p}{\partial y_2}=0$$

Integrals of the system (6) give the dependence of the angle θ on the coordinates q_1 and q_2 and the streamlines on the surface of the body. From Equation (5) we can obtain a nonlinear second order equation for determining the streamlines [3].

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